

On an equation involving fractional powers with one prime and one almost prime variables

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Abstract

In this paper we consider the equation $[p^c] + [m^c] = N$, where N is a sufficiently large integer, and prove that if $1 < c < \frac{29}{28}$, then it has a solution in a prime p and an almost prime m with at most $\left\lfloor \frac{52}{29-28c} \right\rfloor + 1$ prime factors.

1 Introduction and statement of the result

A Piatetski–Shapiro sequence is a sequence of the form

$$\{[n^c]\}_{n \in \mathbb{N}} \quad (c > 1, c \notin \mathbb{N}), \quad (1)$$

where $[t]$ denotes the integer part of t . In 1953 Piatetski–Shapiro [12] showed that if $1 < c < \frac{12}{11}$, then the sequence (1) contains infinitely many prime numbers. Since then, the upper bound for c has been improved many times and the strongest result is due to Rivat and Wu [13]. They proved that the sequence (1) contains infinitely many primes provided that $1 < c < \frac{243}{205}$.

For any natural number r , let \mathcal{P}_r denote the set of r -almost primes, i.e. the set of natural numbers having at most r prime factors counted with multiplicity. There are many papers devoted to the study of problems involving Piatetski–Shapiro primes and almost primes. In 2011 Cai and Wang [2], improving an earlier result of Peneva [11], showed that if $1 < c < \frac{30}{29}$, then there exist infinitely many primes p of the form $[n^c]$ such that $p + 2 \in \mathcal{P}_5$. Later, in 2014, Baker, Banks, Guo, and Yeager [1] showed that if $1 < c < \frac{77}{76}$, then the sequence

$$\{[n^c]\}_{n \in \mathcal{P}_8}$$

contains infinitely many prime numbers.

Consider the equation

$$[m_1^c] + [m_2^c] = N. \quad (2)$$

In 1973, Deshouillers [4] proved that if $1 < c < \frac{4}{3}$, then for every sufficiently large integer N the equation (2) has a solution with m_1 and m_2 integers. This result was improved by Gritsenko [6], and later by Konyagin [9]. In particular, the latter author showed that (2) has a solution in integers m_1, m_2 for $1 < c < \frac{3}{2}$ and N sufficiently large.

Kumchev [10] proved that if $1 < c < \frac{16}{15}$, then every sufficiently large integer N can be represented in the form (2), where m_1 is a prime and m_2 is an integer. On the other hand, the celebrated theorem of Chen [3] states that every sufficiently large even integer can be represented as a sum of a prime and an almost prime from \mathcal{P}_2 . Having in mind this profound result, one can conjecture that there exists a constant $c_0 > 1$ such that if $1 < c < c_0$, then the equation (2) has a solution with m_1 a prime and $m_2 \in \mathcal{P}_2$ provided that N is sufficiently large. In the present paper, we establish a result of this type and prove the following

Theorem 1. *Suppose that $1 < c < \frac{29}{28}$. Then every sufficiently large integer N can be represented as*

$$[p^c] + [m^c] = N, \quad (3)$$

where p is a prime and m is an almost prime with at most $\left[\frac{52}{29-28c}\right] + 1$ prime factors.

We note that the integer $\left[\frac{52}{29-28c}\right] + 1$ is equal to 53 if c is close to 1 and it is large if c is close to $\frac{29}{28}$.

Our first step in the proof is to apply the linear sieve. After doing so, we could try to establish a relatively strong estimate for the exponential sum defined in (31) which is a rather difficult task since the function in the exponent depends on $[p^c]$. Instead, we represent this sum as a linear combination of similar sums (see (60)) with a smooth function of p in the exponent. Then we use standard techniques to estimate these sums. We would like to mention that the sums in (60) are also studied by Kumchev [10]. However, we cannot use his work because we require stronger bounds for them.

2 Notation

We fix the following notation: $\{t\}$ is the fractional part of t , the function $\rho(t)$ is defined by $\rho(t) = \frac{1}{2} - \{t\}$ and $e(t) = e^{2\pi it}$. We use Vinogradov's notation $A \ll B$, which is equivalent to $A = O(B)$. If we have simultaneously $A \ll B$ and $B \ll A$, then we shall write $A \asymp B$.

For us p will be reserved for prime numbers. By ε we denote an arbitrarily small positive number, which is not necessarily the same in the different formulae. As usual, $\sum_{n \leq x}$ means $\sum_{1 \leq n \leq x}$ and $\mu(n)$, $\Lambda(n)$ and $\tau(n)$ are the Mobius function, von Mangoldt's function and the number of positive divisors of n , respectively.

3 Proof of the theorem

3.1 Beginning of the proof

Let N be a sufficiently large integer and let

$$1 < c < \frac{29}{28}, \quad \gamma = \frac{1}{c}, \quad P = 10^{-9}N^\gamma. \quad (4)$$

Suppose that $\alpha > 0$ is a constant, which will be specified later, and let

$$z = N^\alpha, \quad B_z = \prod_{p < z} p. \quad (5)$$

We consider the sum

$$\Gamma = \sum_{\substack{P < p \leq 2P, m \in \mathbb{N} \\ [p^c] + [m^c] = N \\ (m, B_z) = 1}} \log p. \quad (6)$$

If $\Gamma > 0$, then there is a prime number p and a natural number m satisfying the conditions imposed in the domain of summation of Γ . From the condition $(m, B_z) = 1$ it follows that any prime factor of m is greater or equal to z . Suppose that m has l prime factors, counted with the multiplicity. Then we have

$$N^\gamma \geq m \geq z^l = N^{\alpha l}$$

and hence $l \leq \frac{\gamma}{\alpha}$. This implies that if $\Gamma > 0$ then (3) has a solution with p a prime and m an almost prime with at most $\left[\frac{\gamma}{\alpha}\right]$ prime factors.

We denote

$$D = N^\delta, \quad (7)$$

where $\delta > 0$ is a constant which will be specified later. Let $\lambda(d)$ be the lower bound Rosser weights of level D , (see [5, Chapter 4]). Then we have

$$\sum_{d|k} \mu(d) \geq \sum_{d|k} \lambda(d) \quad \text{for every } k \in \mathbb{N}. \quad (8)$$

Furthermore, we know that

$$|\lambda(d)| \leq 1 \quad \text{for all } d; \quad \lambda(d) = 0 \quad \text{if } d > D \quad \text{or} \quad \mu(d) = 0. \quad (9)$$

Finally, we have

$$\sum_{d|B_z} \frac{\lambda(d)}{d} \geq \prod_{p < z} \left(1 - \frac{1}{p}\right) \left(f(s) + O((\log D)^{-\frac{1}{3}})\right), \quad (10)$$

where

$$s = \frac{\log D}{\log z} = \frac{\delta}{\alpha} \quad (11)$$

and where $f(s)$ is the lower function of the linear sieve, for which we know that

$$f(s) = 0 \quad \text{for } 0 < s < 2; \quad f(s) = 2e^G s^{-1} \log(s-1) \quad \text{for } 2 < s < 3. \quad (12)$$

(Here G is the Euler constant).

From (6) and (8) we find

$$\Gamma = \sum_{\substack{P < p \leq 2P, m \in \mathbb{N} \\ [p^c] + [m^c] = N}} (\log p) \sum_{d|(m, B_z)} \mu(d) \geq \sum_{\substack{P < p \leq 2P, m \in \mathbb{N} \\ [p^c] + [m^c] = N}} (\log p) \sum_{d|(m, B_z)} \lambda(d).$$

We change the order of summation to obtain

$$\Gamma \geq \sum_{d|B_z} \lambda(d) G_d, \quad \text{where} \quad G_d = \sum_{\substack{P < p \leq 2P, m \in \mathbb{N} \\ [p^c] + [m^c] = N \\ m \equiv 0 \pmod{d}}} \log p. \quad (13)$$

Now, we write the sum G_d in the form

$$G_d = \sum_{P < p \leq 2P} (\log p) G'_{d,p}, \quad \text{where} \quad G'_{d,p} = \sum_{\substack{m \in \mathbb{N} \\ m \equiv 0 \pmod{d} \\ [p^c] + [m^c] = N}} 1. \quad (14)$$

We use the obvious identity

$$\sum_{a \leq m < b} 1 = [-a] - [-b] = b - a - \rho(-b) + \rho(-a)$$

to establish

$$\begin{aligned} G'_{d,p} &= \sum_{\substack{m \in \mathbb{N} \\ m \equiv 0 \pmod{d} \\ N - [p^c] \leq m^c < N+1 - [p^c]}} 1 = \sum_{\frac{1}{d}(N - [p^c])^\gamma \leq m < \frac{1}{d}(N+1 - [p^c])^\gamma} 1 \\ &= \frac{(N+1 - [p^c])^\gamma - (N - [p^c])^\gamma}{d} + \rho\left(-\frac{1}{d}(N - [p^c])^\gamma\right) - \rho\left(-\frac{1}{d}(N+1 - [p^c])^\gamma\right). \end{aligned}$$

We combine the above with (14) to obtain

$$G_d = \frac{1}{d} A(N) + \sum_{P < p \leq 2P} (\log p) \left(\rho\left(-\frac{1}{d}(N - [p^c])^\gamma\right) - \rho\left(-\frac{1}{d}(N+1 - [p^c])^\gamma\right) \right), \quad (15)$$

where

$$A(N) = \sum_{P < p \leq 2P} (\log p) ((N - [p^c] + 1)^\gamma - (N - [p^c])^\gamma).$$

From

$$(N - [p^c] + 1)^\gamma = (N - [p^c])^\gamma + \gamma(N - [p^c])^{\gamma-1} + O(N^{\gamma-2}),$$

we deduce that

$$A(N) = \gamma \sum_{P < p \leq 2P} (\log p) ((N - [p^c])^{\gamma-1} + O(N^{\gamma-2})),$$

and by Chebyshev's prime number theorem and the definition of P in (4), we get

$$A(N) \asymp N^{2\gamma-1}. \quad (16)$$

From (13) and (15) we have

$$\Gamma \geq \Gamma_0 + \Sigma_0 - \Sigma_1, \quad (17)$$

where

$$\Gamma_0 = A(N) \sum_{d|B_z} \frac{\lambda(d)}{d}, \quad (18)$$

$$\Sigma_j = \sum_{d|B_z} \lambda(d) \sum_{P < p \leq 2P} (\log p) \rho \left(-\frac{1}{d} (N + j - [p^c])^\gamma \right), \quad j = 0, 1. \quad (19)$$

Consider Γ_0 . We use (5) and the Mertens formula to find

$$\prod_{p < z} \left(1 - \frac{1}{p} \right) \asymp \frac{1}{\log z} \asymp \frac{1}{\log N}. \quad (20)$$

Assume that

$$2 < \frac{\delta}{\alpha} < 3. \quad (21)$$

Then, having in mind (11) and (12) we find that $f(s) > \kappa$ for some constant $\kappa > 0$ depending on δ and α only. Therefore, using (10) and (20) we get

$$\sum_{d|B_z} \frac{\lambda(d)}{d} \gg \frac{1}{\log N}.$$

Thus, by (16) and (18) we obtain

$$\Gamma_0 \gg \frac{N^{2\gamma-1}}{\log N}. \quad (22)$$

We aim to establish the following bound for the sums Σ_j defined in (19):

$$\Sigma_j \ll \frac{N^{2\gamma-1}}{(\log N)^2}, \quad j = 0, 1. \quad (23)$$

This, together with (17) and (22) would imply

$$\Gamma \gg \frac{N^{2\gamma-1}}{\log N},$$

hence $\Gamma > 0$ for sufficiently large N . Then, as we already explained, the equation (3) would have a solution in a prime p and an almost prime m with no more than $\left[\frac{\gamma}{\alpha}\right]$ prime factors.

The remaining part of the paper will be devoted to the proof of the estimates (23) under the assumptions

$$\frac{28}{29} < \gamma < 1, \quad \delta < \frac{29\gamma - 28}{26}. \quad (24)$$

Then, it would remain to choose

$$\alpha = \frac{29\gamma - 28}{52} - \varepsilon_0$$

for some small $\varepsilon_0 > 0$ and to take

$$\delta \in \left(2\alpha, \frac{29\gamma - 28}{26}\right).$$

In this case, when ε_0 is small enough the condition (21) holds. With the choice (4) of c , it is easy to see that $\left[\frac{\gamma}{\alpha}\right] \leq \left[\frac{52}{29-28c}\right] + 1$, which proves the theorem.

3.2 The estimation of the sums Σ_1 and Σ_2 — beginning

Consider the sum Σ_j defined in (19). We apply Vaaler's theorem [14], which states that for each $H \geq 2$ there are numbers c_h ($0 < |h| \leq H$), d_h ($|h| \leq H$), such that

$$\rho(t) = \sum_{0 < |h| \leq H} c_h e(ht) + \Delta_H(t), \quad (25)$$

where

$$|\Delta_H(t)| \leq \sum_{|h| \leq H} d_h e(ht) \quad (26)$$

and

$$|c_h| \ll \frac{1}{|h|}, \quad |d_h| \ll \frac{1}{H}. \quad (27)$$

From (19) and (25) it follows that

$$\Sigma_j = \Sigma'_j + \Sigma''_j, \quad (28)$$

where

$$\Sigma'_j = \sum_{d \leq D} \lambda(d) \sum_{P < p \leq 2P} (\log p) \sum_{0 < |h| \leq H} c_h e\left(-\frac{h}{d}(N + j - [p^c])^\gamma\right), \quad (29)$$

$$\Sigma''_j = \sum_{d \leq D} \lambda(d) \sum_{P < p \leq 2P} (\log p) \Delta_H\left(-\frac{(N + j - [p^c])^\gamma}{d}\right). \quad (30)$$

Let

$$W(v) = \sum_{P < p \leq 2P} (\log p) e(v(N + j - [p^c])^\gamma). \quad (31)$$

We start with the sum Σ'_j . Changing the order of summation together with (9), (27) and (31) implies that

$$\Sigma'_j = \sum_{d \leq D} \lambda(d) \sum_{0 < |h| \leq H} c_h W\left(-\frac{h}{d}\right) \ll \sum_{d \leq D} \sum_{1 \leq h \leq H} \frac{1}{h} \left| W\left(\frac{h}{d}\right) \right|, \quad (32)$$

For the sum Σ''_j we use (9), (26), (27) and (31) to get

$$\begin{aligned} \Sigma''_j &\ll \sum_{d \leq D} \sum_{P < p \leq 2P} (\log p) \sum_{|h| \leq H} d_h e\left(-\frac{h}{d}(N + j - [p^c])^\gamma\right) \\ &= \sum_{d \leq D} \sum_{|h| \leq H} d_h W\left(-\frac{h}{d}\right) \ll \sum_{d \leq D} \sum_{|h| \leq H} \frac{1}{H} \left| W\left(\frac{h}{d}\right) \right|. \end{aligned}$$

From (4), (31) and Chebyshev's prime number theorem we find that $W(0) \asymp N^\gamma$ and hence

$$\Sigma''_j \ll \sum_{d \leq D} \frac{N^\gamma}{H} + \sum_{d \leq D} \sum_{1 \leq h \leq H} \frac{1}{H} \left| W\left(\frac{h}{d}\right) \right|. \quad (33)$$

We let

$$H = dN^{1-\gamma}(\log N)^3. \quad (34)$$

Now, using (7), (28) and (32) – (34) we obtain

$$\Sigma_j \ll \frac{N^{2\gamma-1}}{(\log N)^2} + \sum_{d \leq D} \sum_{h \leq H} \frac{1}{h} \left| W\left(\frac{h}{d}\right) \right|, \quad j = 0, 1. \quad (35)$$

3.3 Consideration of the sum $W(v)$

As we mentioned earlier, it is hard to estimate directly the exponential sum $W(v)$, defined by (31). Instead, we can write it as a linear combination of similar sums which are easier to be dealt with.

Let $Z \geq 2$ be an integer, which we shall specify later. We apply the well-known Vinogradov's "little cups" lemma (see [8, Chapter 1, Lemma A]) with parameters

$$\alpha = -\frac{1}{4Z}, \quad \beta = \frac{1}{4Z}, \quad \Delta = \frac{1}{2Z}, \quad r = [\log N]$$

and construct a function $g(t)$ which is periodic with period 1 and has the following properties:

$$g(0) = 1; \quad 0 < g(t) < 1 \quad \text{for} \quad 0 < |t| < \frac{1}{2Z}; \quad g(t) = 0 \quad \text{for} \quad \frac{1}{2Z} \leq |t| \leq \frac{1}{2}. \quad (36)$$

Furthermore, the Fourier series of $g(t)$ is given by

$$g(t) = \frac{1}{2Z} + \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \beta_n e(nt), \quad \text{with} \quad |\beta_n| \leq \min \left(\frac{1}{2Z}, \frac{1}{|n|} \left(\frac{2Z \lfloor \log N \rfloor}{\pi |n|} \right)^{\lfloor \log N \rfloor} \right).$$

From the above estimate of $|\beta_n|$ one easily obtains

$$\sum_{|n| > Z(\log N)^4} |\beta_n| \ll N^{-\log \log N}$$

with an absolute constant in the \ll -symbol. Hence we have

$$g(t) = \sum_{|n| \leq Z(\log N)^4} \beta_n e(nt) + O(N^{-\log \log N}), \quad (37)$$

where the implied constant is absolute and

$$|\beta_n| \leq \frac{1}{2Z}. \quad (38)$$

Finally, one can easily see that the function $g(t)$, constructed in the proof of [8, Chapter 1, Lemma A], is even and also satisfies

$$g(t) + g\left(t - \frac{1}{2Z}\right) = 1 \quad \text{for} \quad 0 \leq t \leq \frac{1}{2Z}. \quad (39)$$

Let

$$g_z(t) = g\left(t - \frac{z}{2Z}\right) \quad \text{for} \quad z = 0, 1, 2, \dots, 2Z - 1. \quad (40)$$

Obviously, each $g_z(t)$ is a periodic function with period 1. From (36) we find that

$$0 < g_z(t) \leq 1 \quad \text{if} \quad \left|t - \frac{z}{2Z}\right| < \frac{1}{2Z}; \quad (41)$$

$$g_z(t) = 0 \quad \text{if} \quad \frac{1}{2Z} \leq \left|t - \frac{z}{2Z}\right| \leq \frac{1}{2}. \quad (42)$$

From (40) it follows that if $\beta_n^{(z)}$ is the n -th Fourier coefficient of the function $g_z(t)$, then $\beta_n^{(z)} = \beta_n e\left(-\frac{zn}{2M}\right)$ and hence $|\beta_n^{(z)}| = |\beta_n|$. From this observation and (37), as well as the estimate for $|\beta_n|$ given above, we find that for $z = 0, 1, \dots, 2Z - 1$ we have

$$g_z(t) = \sum_{|n| \leq Z(\log N)^4} \beta_n^{(z)} e(nt) + O(N^{-\log \log N}), \quad (43)$$

where the constant in the O -symbol is absolute and

$$|\beta_n^{(z)}| \leq \frac{1}{2Z}. \quad (44)$$

Finally, from (39), (40) and (42) we find

$$\sum_{z=0}^{2Z-1} g_z(t) = 1 \quad \text{for all } t \in \mathbb{R}. \quad (45)$$

(We leave the easy verification to the reader).

Now, we consider the sum $W(v)$ which was defined in (31). From (45) it follows that

$$W(v) = \sum_{P < p \leq 2P} (\log p) e(v(N + j - [p^c])^\gamma) \sum_{z=0}^{2Z-1} g_z(p^c) = \sum_{z=0}^{2Z-1} W_z(v), \quad (46)$$

where

$$W_z(v) = \sum_{P < p \leq 2P} (\log p) g_z(p^c) e(v(N + j - [p^c])^\gamma). \quad (47)$$

It is clear that

$$W_0(v) \ll \sum_{P < p \leq 2P} (\log p) g(p^c) \ll \log N \sum_{P < k \leq 2P} g(k^c).$$

We apply (4), (37) and (38) to get

$$\begin{aligned} W_0(v) &\ll (\log N) \frac{P}{Z} + (\log N) \left| \sum_{P < k \leq 2P} \sum_{0 < |n| \leq Z(\log N)^4} \beta_n e(nk^c) \right| + 1 \\ &\ll (\log N) \frac{N^\gamma}{Z} + \frac{\log N}{Z} \sum_{n \leq Z(\log N)^4} |\mathcal{H}_n| + 1, \end{aligned} \quad (48)$$

where

$$\mathcal{H}_n = \sum_{P < k \leq 2P} e(nk^c).$$

If $\theta(x) = nx^c$, then $\theta''(x) = c(c-1)nx^{c-2} \asymp nP^{c-2}$ uniformly for $x \in [P, 2P]$. Hence, we can apply Van der Corput's theorem (see [8], Chapter 1, Theorem 5) to get

$$\mathcal{H}_n \ll P (nP^{c-2})^{\frac{1}{2}} + (nP^{c-2})^{-\frac{1}{2}} \ll P^{\frac{c}{2}} n^{\frac{1}{2}}. \quad (49)$$

Henceforth we assume that

$$Z \ll N^{\frac{2\gamma-1}{3}} (\log N)^{-4}. \quad (50)$$

Then using (4) and (48) – (50) we obtain

$$W_0(v) \ll (\log N) \left(\frac{N^\gamma}{Z} + N^{\frac{1}{2}} Z^{\frac{1}{2}} \log^6 N \right) \ll (\log N) \frac{N^\gamma}{Z}. \quad (51)$$

Now, we restrict our attention to the sums $W_z(v)$ for $1 \leq z \leq 2Z - 1$. Using (42) we see that $g_z(p^c)$ vanishes unless $\{p^c\} \in [\frac{z-1}{2Z}, \frac{z+1}{2Z}]$. Hence, the only summands in the sum (47) are those for which $\{p^c\} = \frac{z}{2Z} + O(\frac{1}{Z})$. In this case we have

$$v(N + j - [p^c])^\gamma = v\left(N + j - p^c + \frac{z}{2Z}\right)^\gamma + O\left(\frac{vN^{\gamma-1}}{Z}\right)$$

and respectively

$$e(v(N + j - [p^c])^\gamma) = e\left(v\left(N + j - p^c + \frac{z}{2Z}\right)^\gamma\right) + O\left(\frac{vN^{\gamma-1}}{Z}\right).$$

Then using (47) we find

$$W_z(v) = V_z(v) + O\left(\frac{vN^{\gamma-1}}{Z} \sum_{P < p \leq 2P} (\log p) g_m(p^c)\right), \quad (52)$$

where

$$V_z(v) = \sum_{P < p \leq 2P} (\log p) g_z(p^c) e\left(v\left(N + j - p^c + \frac{z}{2Z}\right)^\gamma\right). \quad (53)$$

Therefore, from (46), (51) and (52) we obtain

$$W(v) = \sum_{z=1}^{2Z-1} W_z(v) + W_0(v) = \sum_{z=1}^{2Z-1} V_z(v) + O(\Xi) + O\left((\log N) \frac{N^\gamma}{Z}\right),$$

where

$$\Xi = \frac{vN^{\gamma-1}}{Z} \sum_{P < p \leq 2P} (\log p) \sum_{z=1}^{2Z-1} g_z(p^c).$$

Now we use (4), (45) and Chebyshev's prime number theorem to find that

$$\Xi \ll \frac{vN^{2\gamma-1}}{Z}$$

and therefore

$$W(v) = \sum_{z=1}^{2Z-1} V_z(v) + O\left(\frac{vN^{2\gamma-1}}{Z}\right) + O\left((\log N) \frac{N^\gamma}{Z}\right). \quad (54)$$

From this point onwards we assume that

$$v = \frac{h}{d}, \quad \text{where} \quad 1 \leq d \leq D, \quad 1 \leq h \leq H. \quad (55)$$

Then using (34) we see that $vN^{2\gamma-1} \ll N^\gamma(\log N)^3$, hence formula (54) can be written as

$$W(v) = \sum_{z=1}^{2Z-1} V_z(v) + O\left((\log N)^3 \frac{N^\gamma}{Z}\right). \quad (56)$$

From (34), (35) and (56) we find

$$\Sigma_j \ll \frac{N^{2\gamma-1}}{(\log N)^2} + \sum_{d \leq D} \sum_{h \leq H} \frac{1}{h} \sum_{z=1}^{2Z-1} |V_z(v)| + \sum_{d \leq D} \sum_{h \leq H} \frac{1}{h} \frac{N^\gamma}{Z} (\log N)^3. \quad (57)$$

We choose Z such that

$$Z \asymp dN^{1-\gamma}(\log N)^7. \quad (58)$$

From (7) and (24) it follows that the condition (50) holds. Consequently from (57) and (58) we find

$$\Sigma_j \ll \frac{N^{2\gamma-1}}{(\log N)^2} + \sum_{d \leq D} \sum_{h \leq H} \frac{1}{h} \sum_{z=1}^{2Z-1} |V_z(v)|. \quad (59)$$

Now, we consider the sum $V_z(v)$ defined in (53), where v satisfies (55). By (43) we find that

$$\begin{aligned} V_z(v) &= \sum_{P < p \leq 2P} (\log p) \left(\sum_{|r| \leq Z(\log N)^4} \beta_r^{(z)} e(rp^c) \right) e\left(v \left(N + j - p^c + \frac{z}{2Z}\right)^\gamma\right) + O(N^{-10}) \\ &= \sum_{|r| \leq Z(\log N)^4} \beta_r^{(z)} U\left(N + j + \frac{z}{2Z}, r, v\right) + O(N^{-10}), \end{aligned}$$

where

$$U = U(T, r, v) = \sum_{P < p \leq 2P} (\log p) e(rp^c + v(T - p^c)^\gamma). \quad (60)$$

We would like to point out that when $0 \leq j \leq 1$ and $1 \leq z \leq 2Z - 1$ then

$$N + j + \frac{z}{2Z} \in [N, N + 2].$$

Furthermore, when we also take into account (44) and (58) we obtain

$$V_z(v) \ll N^{-10} + \frac{1}{Z} \sum_{|r| \leq R} \sup_{T \in [N, N+2]} |U(T, r, v)|.$$

where

$$R = dN^{1-\gamma}(\log N)^{12} \quad (61)$$

When we substitute the above expression for $V_z(v)$ in formula (59) we find that

$$|\Sigma_1| + |\Sigma_2| \ll \frac{N^{2\gamma-1}}{(\log N)^2} + \sum_{d \leq D} \sum_{h \leq H} \frac{1}{h} \sum_{|r| \leq R} \sup_{T \in [N, N+2]} |U(T, r, v)|. \quad (62)$$

3.4 Application of Vaughan's identity

Let us introduce the notations

$$\phi(t) = rt^c + v(T - t^c)^\gamma, \quad (63)$$

$$f(m, l) = \phi(ml) = r(ml)^c + v(T - (ml)^c)^\gamma. \quad (64)$$

For the sum U , defined in (60), we have

$$U = \sum_{P < n \leq 2P} \Lambda(n) e(\phi(n)) + O\left(P^{\frac{1}{2}}\right).$$

Now we apply Vaughan's identity (see [15]) and find that

$$U = U_1 - U_2 - U_3 - U_4 + O\left(P^{\frac{1}{2}}\right), \quad (65)$$

where

$$U_1 = \sum_{m \leq P^{\frac{1}{3}}} \mu(m) \sum_{\frac{P}{m} < l \leq \frac{2P}{m}} (\log l) e(f(m, l)), \quad (66)$$

$$U_2 = \sum_{m \leq P^{\frac{1}{3}}} c(m) \sum_{\frac{P}{m} < l \leq \frac{2P}{m}} e(f(m, l)), \quad (67)$$

$$U_3 = \sum_{P^{\frac{1}{3}} < m \leq P^{\frac{2}{3}}} c(m) \sum_{\frac{P}{m} < l \leq \frac{2P}{m}} e(f(m, l)), \quad (68)$$

$$U_4 = \sum_{\substack{P < ml \leq 2P \\ m > P^{\frac{1}{3}}, l > P^{\frac{1}{3}}}} a(m) \Lambda(l) e(f(m, l)), \quad (69)$$

and where

$$|c(m)| \leq \log m \quad \text{and} \quad |a(m)| \leq \tau(m). \quad (70)$$

Hence from (7), (24), (34), (61), (62) and (65) we have

$$|\Sigma_1| + |\Sigma_2| \ll \frac{N^{2\gamma-1}}{(\log N)^2} + \sum_{i=1}^4 \Omega_i, \quad (71)$$

where

$$\Omega_i = \sum_{d \leq D} \sum_{h \leq H} \frac{1}{h} \sum_{|r| \leq R} \sup_{T \in [N, N+2]} |U_i|. \quad (72)$$

By (71) and (72), in order to prove that (23) is satisfied, it suffices to show that

$$\Omega_i \ll \frac{N^{2\gamma-1}}{(\log N)^2} \quad \text{for} \quad i = 1, 2, 3, 4. \quad (73)$$

3.5 The estimation of the sums Ω_1 and Ω_2

We begin with the study of Ω_2 . From (64) we get

$$f''_{ll}(m, l) = \pi_1 - \pi_2, \quad (74)$$

where

$$\pi_1 = m^2 r c (c-1) (ml)^{c-2}, \quad \pi_2 = m^2 v (c-1) T (ml)^{c-2} (T - (ml)^c)^{\gamma-2}. \quad (75)$$

From (4), (55) and the conditions

$$P < ml \leq 2P, \quad N \leq T \leq N+2 \quad (76)$$

we have

$$|\pi_1| \asymp |r| m^2 N^{1-2\gamma} \quad \text{and} \quad \pi_2 \asymp v m^2 N^{-\gamma}. \quad (77)$$

It follows from (74) and (77) that there exists sufficiently small constant $\alpha_0 > 0$ such that if $|r| \leq \alpha_0 v N^{\gamma-1}$, then $|f''_{ll}| \asymp v m^2 N^{-\gamma}$.

Similarly, from (74) and (77) we conclude that there exists sufficiently large constant $A_0 > 0$ such that if $|r| \geq A_0 v N^{\gamma-1}$, then $|f''_{ll}| \asymp |r| m^2 N^{1-2\gamma}$.

We divide the sum Ω_2 into four sums according to the value of r as follows:

$$\Omega_2 = \Omega_{2,1} + \Omega_{2,2} + \Omega_{2,3} + \Omega_{2,4}, \quad (78)$$

where

$$\text{in } \Omega_{2,1} : \quad |r| \leq \alpha_0 v N^{\gamma-1}, \quad (79)$$

$$\text{in } \Omega_{2,2} : \quad -A_0 v N^{\gamma-1} < r < -\alpha_0 v N^{\gamma-1}, \quad (80)$$

$$\text{in } \Omega_{2,3} : \quad \alpha_0 v N^{\gamma-1} < r < A_0 v N^{\gamma-1}, \quad (81)$$

$$\text{in } \Omega_{2,4} : \quad A_0 v N^{\gamma-1} \leq |r| \leq R. \quad (82)$$

We note that from (34), (55) and (61) it follows that

$$v N^{\gamma-1} \ll (\log N)^3 \ll \frac{R}{\log N}. \quad (83)$$

Let us consider $\Omega_{2,4}$ first. We have

$$\Omega_{2,4} = \sum_{d \leq D} \sum_{h \leq H} \frac{1}{h} \sum_{A_0 v N^{\gamma-1} \leq |r| \leq R} \sup_{T \in [N, N+2]} |U_2|. \quad (84)$$

We shall estimate the sum U_2 , defined by (67), provided that the condition (82) holds. We recall that the constant A_0 is chosen such a way, that if $|r| \geq A_0 v N^{\gamma-1}$, then uniformly for $l \in (\frac{P}{m}, \frac{2P}{m}]$ we have $|f''_{ll}(m, l)| \asymp |r| m^2 N^{1-2\gamma}$. Hence we are in a position to use again Van der Corput's theorem (see [8, Chapter 1, Theorem 5]) and having also in mind (4) we obtain

$$\sum_{\frac{P}{m} < l \leq \frac{2P}{m}} e(f(m, l)) \ll \frac{P}{m} (|r| m^2 N^{1-2\gamma})^{\frac{1}{2}} + (|r| m^2 N^{1-2\gamma})^{-\frac{1}{2}} \ll |r|^{\frac{1}{2}} N^{\frac{1}{2}}.$$

Then from (67) and (70) we find

$$U_2 \ll |r|^{\frac{1}{2}} N^{\frac{1}{2} + \frac{\gamma}{3}} (\log N). \quad (85)$$

We substitute this expression for U_2 in (84) and use (34), (55) and (61) to get

$$\Omega_{2,4} \ll D^{\frac{5}{2}} N^{2 - \frac{7\gamma}{6} + \varepsilon}.$$

Hence from (7) and (24) we obtain

$$\Omega_{2,4} \ll \frac{N^{2\gamma-1}}{(\log N)^2}. \quad (86)$$

We carry on with $\Omega_{2,3}$. We have to study the sum U_2 defined by (67) provided that the condition (81) holds. To do so we use (64) and compute

$$\begin{aligned} f'''_{lll}(m, l) &= m^3 r c(c-1)(c-2)(ml)^{c-3} \\ &\quad + m^3 v(c-1)T(T - (ml)^c)^{\gamma-3} (ml)^{c-3} ((2-c)T - (c+1)(ml)^c). \end{aligned} \quad (87)$$

From (74), (75) and (87) we find

$$(c-2)f''_{ll}(m, l) - l f'''_{lll}(m, l) = v(c-1)(2c-1)T m^{2c} l^{2c-2} (T - (ml)^c)^{\gamma-3}.$$

The above, together with (4) and (76) implies

$$|(c-2)f''_{ll}(m, l) - l f'''_{lll}(m, l)| \asymp v m^2 N^{-\gamma}.$$

Therefore, there exists $\kappa_0 > 0$, depending only on the constant c , such that for every $l \in (\frac{P}{m}, \frac{2P}{m}]$ at least one of the following inequalities holds:

$$|f''_{ll}(m, l)| \geq \kappa_0 v m^2 N^{-\gamma}, \quad (88)$$

or

$$|f'''_{lll}(m, l)| \geq \kappa_0 v m^3 N^{-2\gamma}. \quad (89)$$

We are going to show that the interval $(\frac{P}{m}, \frac{2P}{m}]$ can be divided into at most 7 intervals such that if J is one of them, then at least one of the following statements holds:

$$\text{We have (88) for all } l \in J. \quad (90)$$

We have (89) for all $l \in J$. (91)

To establish this it is enough to show that the equation $|f''_{ll}(m, l)| = \kappa_0 v m^2 N^{-\gamma}$ has at most 6 solutions in real numbers $l \in (\frac{P}{m}, \frac{2P}{m})$. Hence, it is enough to show that if C does not depend on l then the equation $f''_{ll}(m, l) = C$ has at most 3 solutions in real numbers $l \in (\frac{P}{m}, \frac{2P}{m})$. According to Rolle's theorem, between any two solutions of the last equation there is a solution (in real numbers l) of the equation $f'''_{lll}(m, l) = 0$. So, we use (87) to conclude that it is enough to show that

$$vT(T - (ml)^c)^{\gamma-3}((2-c)T - (c+1)(ml)^c) = rc(2-c)$$

has at most 2 solutions in $l \in (\frac{P}{m}, \frac{2P}{m})$, which is equivalent to the assertion that the equation

$$(T - X)^{\gamma-3}((2-c)T - (c+1)X) = \frac{rc(2-c)}{vT}$$

has at most 2 solutions in $X \in (P^c, (2P)^c)$. Alternatively, instead of the last equation one can look at

$$(\gamma - 3) \log(T - X) + \log((2-c)T - (c+1)X) = \log \frac{rc(2-c)}{vT}. \quad (92)$$

Let $H(X)$ denote the function on the left side of (92). From Rolle's theorem we know that between any two solutions of (92) there is a solution of $H'(X) = 0$. Since

$$H'(X) = \frac{3-\gamma}{T-X} - \frac{c+1}{(2-c)T - (c+1)X}$$

it is easy to see that $H'(X)$ vanishes for at most 1 value of X . Therefore, (92) has at most 2 solutions in X and our assertion is proved.

On the other hand, from (74), (77) and (87) we see that under the condition on r imposed in (81) we have

$$f''_{ll}(m, l) \ll v m^2 N^{-\gamma} \quad \text{and} \quad f'''_{lll}(m, l) \ll v m^3 N^{-2\gamma}.$$

Hence, the interval $(\frac{P}{m}, \frac{2P}{m}]$ can be divided into at most 7 intervals such that if J is one of them, then at least one of the following assertions holds:

$$|f''_{ll}(m, l)| \asymp v m^2 N^{-\gamma} \quad \text{uniformly for} \quad l \in J, \quad (93)$$

$$|f'''_{lll}(m, l)| \asymp v m^3 N^{-2\gamma} \quad \text{uniformly for} \quad l \in J. \quad (94)$$

If (93) is fulfilled, then we use Van der Corput's theorem (see [8, Chapter 1, Theorem 5]) for the second derivative and find

$$\sum_{l \in J} e(f(m, l)) \ll \frac{P}{m} (v m^2 N^{-\gamma})^{\frac{1}{2}} + (v m^2 N^{-\gamma})^{-\frac{1}{2}} \ll v^{\frac{1}{2}} N^{\frac{\gamma}{2}} + v^{-\frac{1}{2}} m^{-1} N^{\frac{\gamma}{2}}. \quad (95)$$

In the case (94) we apply Van der Corput's theorem for the third derivative to get

$$\sum_{l \in J} e(f(m, l)) \ll \frac{P}{m} (vm^3 N^{-2\gamma})^{\frac{1}{6}} + \left(\frac{P}{m}\right)^{\frac{1}{2}} (vm^3 N^{-2\gamma})^{-\frac{1}{6}} \ll v^{\frac{1}{6}} m^{-\frac{1}{2}} N^{\frac{2\gamma}{3}} + v^{-\frac{1}{6}} m^{-1} N^{\frac{5\gamma}{6}}.$$

Hence, in either case $\sum_{l \in J} e(f(m, l))$ can be estimated by the sum of the expressions on the right sides of the inequalities above. Therefore,

$$\sum_{\frac{P}{m} < l \leq \frac{2P}{m}} e(f(m, l)) \ll v^{\frac{1}{2}} N^{\frac{\gamma}{2}} + v^{-\frac{1}{2}} m^{-1} N^{\frac{\gamma}{2}} + v^{\frac{1}{6}} m^{-\frac{1}{2}} N^{\frac{2\gamma}{3}} + v^{-\frac{1}{6}} m^{-1} N^{\frac{5\gamma}{6}}. \quad (96)$$

Then from (4), (67) and (70) we find that

$$U_2 \ll (\log N)^2 \left(v^{\frac{1}{2}} N^{\frac{5\gamma}{6}} + v^{-\frac{1}{2}} N^{\frac{\gamma}{2}} + v^{\frac{1}{6}} N^{\frac{5\gamma}{6}} + v^{-\frac{1}{6}} N^{\frac{5\gamma}{6}} \right). \quad (97)$$

We use (34), (55), (81), (83) and (97) to get

$$\begin{aligned} \Omega_{2,3} &= \sum_{d \leq D} \sum_{h \leq H} \frac{1}{h} \sum_{\alpha_0 v N^{\gamma-1} < r < A_0 v N^{\gamma-1}} \sup_{T \in [N, N+2]} |U_2| \\ &\ll N^\varepsilon \sum_{d \leq D} \sum_{h \leq H} \frac{1}{h} \left(\frac{h^{\frac{1}{2}} N^{\frac{5\gamma}{6}}}{d^{\frac{1}{2}}} + \frac{h^{-\frac{1}{2}} N^{\frac{\gamma}{2}}}{d^{-\frac{1}{2}}} + \frac{h^{\frac{1}{6}} N^{\frac{5\gamma}{6}}}{d^{\frac{1}{6}}} + \frac{h^{-\frac{1}{6}} N^{\frac{5\gamma}{6}}}{d^{-\frac{1}{6}}} \right) \\ &\ll N^\varepsilon \left(D N^{\frac{1}{2} + \frac{\gamma}{3}} + D^{\frac{3}{2}} N^{\frac{\gamma}{2}} + D N^{\frac{1}{6} + \frac{2\gamma}{3}} + D^{\frac{7}{6}} N^{\frac{5\gamma}{6}} \right) \end{aligned} \quad (98)$$

and from (7) and (24) we deduce that

$$\Omega_{2,3} \ll \frac{N^{2\gamma-1}}{(\log N)^2}. \quad (99)$$

Let us consider $\Omega_{2,1}$. We have chosen the constant α_0 in such a way, that from (76) and from the condition on r imposed in (79) it follows that $|f''_{ll}(m, l)| \asymp vm^2 N^{-\gamma}$ uniformly for $l \in (\frac{P}{m}, \frac{2P}{m}]$. Hence the sum $\sum_{\frac{P}{m} < l \leq \frac{2P}{m}} e(f(m, l))$ can be estimated by the expression on the right side of (95) and certainly the estimate (96) holds again. From this observation we see that $\Omega_{2,1}$ can be estimated in the same way as $\Omega_{2,3}$, i.e.

$$\Omega_{2,1} \ll \frac{N^{2\gamma-1}}{(\log N)^2}. \quad (100)$$

The sum $\Omega_{2,2}$ can be studied in the same way. From (74) – (76) and (80) it follows that $|f''_{ll}(m, l)| \asymp vm^2 N^{-\gamma}$ and hence the estimate (96) is correct again. Therefore

$$\Omega_{2,2} \ll \frac{N^{2\gamma-1}}{(\log N)^2}. \quad (101)$$

From (78), (86) and (99) – (101) we conclude that

$$\Omega_2 \ll \frac{N^{2\gamma-1}}{(\log N)^2}. \quad (102)$$

Consider now Ω_1 . For U_1 defined by (66), we use Abel's transformation to get rid of the factor $\log l$ in the inner sum. Then we proceed as in the estimation of Ω_2 to obtain

$$\Omega_1 \ll \frac{N^{2\gamma-1}}{(\log N)^2}. \quad (103)$$

3.6 The estimation of the sums Ω_3 and Ω_4 and the end of the proof

Consider the sum Ω_4 , defined in (72). We divide the sum U_4 given by (69) into $O(\log N)$ sums of the form

$$W_{M,L} = \sum_{L < l \leq 2L} b(l) \sum_{\substack{M < m \leq 2M \\ \frac{P}{l} < m \leq \frac{2P}{l}}} a(m) e(f(m, l)), \quad (104)$$

where

$$a(m) \ll N^\varepsilon, \quad b(l) \ll N^\varepsilon, \quad P^{\frac{1}{3}} \leq M \leq P^{\frac{1}{2}} \ll L \ll P^{\frac{2}{3}}, \quad ML \asymp P. \quad (105)$$

From (104), (105) and Cauchy's inequality we find that

$$|W_{M,L}|^2 \ll N^\varepsilon L \sum_{L < l \leq 2L} \left| \sum_{M_1 < m \leq M_2} a(m) e(f(m, l)) \right|^2, \quad (106)$$

where

$$M_1 = \max \left(M, \frac{P}{l} \right), \quad M_2 = \min \left(2M, \frac{2P}{l} \right). \quad (107)$$

Now we apply the well-known inequality

$$\left| \sum_{a < m \leq b} \xi(m) \right|^2 \leq \frac{b-a+Q}{Q} \sum_{|q| < Q} \left(1 - \frac{|q|}{Q} \right) \sum_{\substack{m \in (a, b] \\ m \in (a-q, b-q]}} \xi(m+q) \overline{\xi(m)}, \quad (108)$$

where $Q \in \mathbb{N}$, $a, b \in \mathbb{R}$, $1 \leq b-a$ and $\xi(m)$ is any complex function. (A proof can be found in [7, Lemma 8.17]). In our setting $\xi(m) = a(m) e(f(m, l))$, $a = M_1$, $b = M_2$. The exact value of Q will be chosen later. For now we only require that

$$Q \leq M. \quad (109)$$

Then we find

$$|W_{M,L}|^2 \ll N^\varepsilon L \sum_{L < l \leq 2L} \frac{M}{Q} \sum_{|q| \leq Q} \left(1 - \frac{|q|}{Q}\right) \\ \times \sum_{\substack{M_1 < m \leq M_2 \\ M_1 < m+q \leq M_2}} a(m+q) \overline{a(m)} e(f(m+q, l) - f(m, l)).$$

We estimate the contribution coming from the terms with $q = 0$, then we change the order of summation and using (105) and (107) we find

$$|W_{M,L}|^2 \ll \frac{N^\varepsilon (LM)^2}{Q} + \frac{N^\varepsilon LM}{Q} \sum_{0 < |q| \leq Q} \sum_{\substack{M < m \leq 2M \\ M < m+q \leq 2M}} \left| \sum_{L_1 < l \leq L_2} e(Y_{m,q}(l)) \right|, \quad (110)$$

where

$$L_1 = \max \left(L, \frac{P}{m}, \frac{P}{m+q} \right), \quad L_2 = \min \left(2L, \frac{2P}{m}, \frac{2P}{m+q} \right) \quad (111)$$

and

$$Y(l) = Y_{m,q}(l) = f(m+q, l) - f(m, l). \quad (112)$$

It is now easy to see that the sum over negative q in formula (110) is equal to the sum over positive q , hence we obtain

$$|W_{M,L}|^2 \ll \frac{N^\varepsilon (LM)^2}{Q} + \frac{N^\varepsilon LM}{Q} \sum_{1 \leq q \leq Q} \sum_{M < m \leq 2M-q} \left| \sum_{L_1 < l \leq L_2} e(Y_{m,q}(l)) \right|. \quad (113)$$

Consider the function $Y(l)$. Using (63), (64) and (112) we find that

$$Y(l) = \int_m^{m+q} f'_t(t, l) dt = \int_m^{m+q} l \phi'(tl) dt \quad (114)$$

and therefore

$$Y''(l) = \int_m^{m+q} (2t \phi''(tl) + lt^2 \phi'''(tl)) dt, \quad Y'''(l) = \int_m^{m+q} (3t^2 \phi'''(tl) + lt^3 \phi^{(4)}(tl)) dt. \quad (115)$$

From (63) and (115) we get

$$Y''(l) = \int_m^{m+q} (\Phi_1(t) - \Phi_2(t)) dt, \quad (116)$$

where

$$\Phi_1(t) = rc^2(c-1)t^{c-1}l^{c-2}, \quad (117)$$

$$\Phi_2(t) = v(c-1)Tt^{c-1}l^{c-2}(T - (tl)^c)^{\gamma-3}(cT + (c-1)(tl)^c). \quad (118)$$

If $t \in [m, m+q]$ then $tl \asymp P$. Thus, by (4) and the condition

$$N \leq T \leq N+2 \quad (119)$$

we find that uniformly for $t \in [m, m+q]$ we have

$$|\Phi_1(t)| \asymp |r| m N^{1-2\gamma} \quad \text{and} \quad \Phi_2(t) \asymp v m N^{-\gamma}. \quad (120)$$

From (116) and (120) we see that there exists a sufficiently small constant $\alpha_1 > 0$ such that if $|r| \leq \alpha_1 v N^{\gamma-1}$, then $|Y''(l)| \asymp qvmN^{-\gamma}$. Similarly, we conclude that there exists a sufficiently large constant $A_1 > 0$ such that if $|r| \geq A_1 v N^{\gamma-1}$, then $|Y''(l)| \asymp |r| qm N^{1-2\gamma}$. Hence, it makes sense to divide the sum Ω_4 into four sums according to the value of r as follows:

$$\Omega_4 = \Omega_{4,1} + \Omega_{4,2} + \Omega_{4,3} + \Omega_{4,4}, \quad (121)$$

where

$$\text{in } \Omega_{4,1} : \quad |r| \leq \alpha_1 v N^{\gamma-1}, \quad (122)$$

$$\text{in } \Omega_{4,2} : \quad -A_1 v N^{\gamma-1} < r < -\alpha_1 v N^{\gamma-1}, \quad (123)$$

$$\text{in } \Omega_{4,3} : \quad \alpha_1 v N^{\gamma-1} < r < A_1 v N^{\gamma-1}, \quad (124)$$

$$\text{in } \Omega_{4,4} : \quad A_1 v N^{\gamma-1} \leq |r| \leq R. \quad (125)$$

Let us consider $\Omega_{4,4}$ first. From (72) and (125) we have

$$\Omega_{4,4} \ll (\log N) \sum_{d \leq D} \sum_{h \leq H} \frac{1}{h} \sum_{A_1 v N^{\gamma-1} \leq |r| \leq R} \sup_{\substack{T \in [N, N+2] \\ M, L : (105)}} |W_{M,L}|. \quad (126)$$

(The supremum is taken over $T \in [N, N+2]$ and M, L satisfying the conditions imposed in (105)).

Consider the sum $W_{M,L}$. We already mentioned that if $|r| \geq A_1 v N^{\gamma-1}$, then uniformly for $l \in (L_1, L_2]$ we have $Y''(l) \asymp |r| qm N^{1-2\gamma}$. Hence we can use Van der Corput's theorem (see [8, Chapter 1, Theorem 5]) for the second derivative and by (4), (105) and (111) we obtain

$$\sum_{L_1 < l \leq L_2} e(Y(l)) \ll L(|r| qm N^{1-2\gamma})^{\frac{1}{2}} + (|r| qm N^{1-2\gamma})^{-\frac{1}{2}} \ll |r|^{\frac{1}{2}} q^{\frac{1}{2}} M^{-\frac{1}{2}} N^{\frac{1}{2}}.$$

Then from (4), (105) and (113) we find

$$W_{M,L} \ll N^\varepsilon \left(N^\gamma Q^{-\frac{1}{2}} + |r|^{\frac{1}{4}} Q^{\frac{1}{4}} N^{\frac{1}{4} + \frac{5\gamma}{8}} \right). \quad (127)$$

From (34), (61) and (126) we have

$$\begin{aligned} \Omega_{4,4} &\ll N^\varepsilon \sum_{d \leq D} \sum_{h \leq H} \frac{1}{h} \sum_{|r| \leq dN^{1-\gamma}(\log N)^{12}} \left(N^\gamma Q^{-\frac{1}{2}} + |r|^{\frac{1}{4}} Q^{\frac{1}{4}} N^{\frac{1}{4} + \frac{5\gamma}{8}} \right) \\ &\ll N^\varepsilon \left(D^2 N Q^{-\frac{1}{2}} + Q^{\frac{1}{4}} D^{\frac{9}{4}} N^{\frac{3}{2} - \frac{5\gamma}{8}} \right). \end{aligned}$$

We choose

$$Q = \left[D^{-\frac{1}{3}} N^{\frac{5\gamma}{6} - \frac{2}{3}} \right]. \quad (128)$$

It is now easy to verify that the condition (109) holds. Hence, from (7) and (24) we obtain

$$\Omega_{4,4} \ll \frac{N^{2\gamma-1}}{(\log N)^2}. \quad (129)$$

Let us now consider $\Omega_{4,3}$. From (72) and (124) we have

$$\Omega_{4,3} \ll (\log N) \sum_{d \leq D} \sum_{h \leq H} \frac{1}{h} \sum_{\alpha_1 v N^{\gamma-1} < r < A_1 v N^{\gamma-1}} \sup_{\substack{T \in [N, N+2] \\ M, L : (105)}} |W_{M,L}|. \quad (130)$$

Consider the sum $W_{M,L}$ from the expression in the above formula. Using (116) – (118) we find

$$Y'''(l) = \int_m^{m+q} (\Psi_1(t) + \Psi_2(t)) dt, \quad (131)$$

where

$$\Psi_1(t) = r c^2 (c-1) (c-2) t^{c-1} l^{c-3}, \quad (132)$$

$$\begin{aligned} \Psi_2(t) &= v (c-1) T t^{c-1} l^{c-3} (T - (tl)^c)^{\gamma-4} \\ &\quad \times \left(c(2-c) T^2 + (-4c^2 + 3c - 2) T (tl)^c + (1 - c^2) (tl)^{2c} \right). \end{aligned} \quad (133)$$

From (116) – (118) and (131) – (133) we obtain

$$lY'''(l) + (2-c)Y''(l) = - \int_m^{m+q} \Theta(t) dt,$$

where

$$\Theta(t) = v(c-1) T t^{2c-1} l^{2c-2} (T - (tl)^c)^{\gamma-4} (2c(2c-1)T + (2c^2 - 3c + 1)(tl)^c).$$

Using (4), (105) and (119) we find that uniformly for $t \in [m, m+q]$ we have

$$\Theta(t) \asymp vmN^{-\gamma}.$$

Hence

$$|(2-c)Y''(l) + lY'''(l)| \asymp qmvN^{-\gamma}$$

uniformly for $l \in (L_1, L_2]$. Therefore, there exists $\kappa_1 > 0$ which depends only on c and such that, at least one of the following inequalities holds for every $l \in (L_1, L_2]$:

$$|Y''(l)| \geq \kappa_1 vqmN^{-\gamma}, \quad (134)$$

$$|Y'''(l)| \geq \kappa_1 vqm^2N^{-2\gamma}. \quad (135)$$

The next step is to show that the interval $(L_1, L_2]$ can be divided into at most 13 intervals such that if J is one of them, then at least one of the following assertions holds:

$$\text{We have (134) for all } l \in J. \quad (136)$$

$$\text{We have (135) for all } l \in J. \quad (137)$$

To establish this, it suffices to show that the equation $|Y''(l)| = \kappa_1 vqmN^{-\gamma}$ has at most 12 solutions in real numbers $l \in (L_1, L_2)$. Hence, it is enough to show that if C does not depend on l , then the equation $Y''(l) = C$ has at most 6 solutions in real numbers $l \in (L_1, L_2)$. According to Rolle's theorem, between any two solutions of the last equation there is a solution of the equation

$$Y'''(l) = 0. \quad (138)$$

Hence, it is enough to show that (138) has at most 5 solutions in real numbers $l \in (L_1, L_2)$.

By (87) and (112) one can easily see that (138) is equivalent to the equation

$$\begin{aligned} & (m+q)^c (T - (m+q)^c l^c)^{\gamma-3} ((2-c)T - (c+1)(m+q)^c l^c) \\ & - m^c (T - m^c l^c)^{\gamma-3} ((2-c)T - (c+1)m^c l^c) = \frac{rc(2-c)((m+q)^c - m^c)}{vT}. \end{aligned}$$

Let $X = l^c$. Define

$$\begin{aligned} \mathcal{F}(X) &= (m+q)^c (T - (m+q)^c X)^{\gamma-3} ((2-c)T - (c+1)(m+q)^c X) \\ & - m^c (T - m^c X)^{\gamma-3} ((2-c)T - (c+1)m^c X). \end{aligned}$$

It would be enough to show that if B does not depend on X , then the equation $\mathcal{F}(X) = B$ has at most 5 solutions with $X \in (L_1^c, L_2^c)$.

Once more, we refer to Rolle's theorem to justify that it is enough to prove that the equation $\mathcal{F}'(X) = 0$ has no more than 4 solutions with $X \in (L_1^c, L_2^c)$. One could write $\mathcal{F}'(X) = 0$ as

$$\begin{aligned} & (m+q)^{2c}(T - (m+q)^c X)^{\gamma-4}((4c+2\gamma-6)T + (2c-\gamma+1)(m+q)^c X) \\ &= m^{2c}(T - m^c X)^{\gamma-4}((4c+2\gamma-6)T + (2c-\gamma+1)m^c X), \end{aligned}$$

which, in turn, is equivalent to

$$\mathcal{G}(X) = \log m^c - \log(m+q)^c,$$

where

$$\begin{aligned} \mathcal{G}(X) &= (\gamma-4) \log(T - (m+q)^c X) + \log((4c+2\gamma-6)T + (2c+1-\gamma)(m+q)^c X) \\ &\quad - (\gamma-4) \log(T - m^c X) - \log((4c+2\gamma-6)T + (2c+1-\gamma)m^c X). \end{aligned}$$

By the same argument as before it is enough to establish that the equation

$$\mathcal{G}'(X) = 0 \tag{139}$$

has at most 3 solutions with $X \in (L_1^c, L_2^c)$. This can easily be shown because

$$\begin{aligned} \mathcal{G}'(X) &= \frac{(4-\gamma)(m+q)^c}{T - (m+q)^c X} + \frac{(2c+1-\gamma)(m+q)^c}{(4c+2\gamma-6)T + (2c+1-\gamma)(m+q)^c X} \\ &\quad - \frac{(4-\gamma)m^c}{T - m^c X} - \frac{(2c+1-\gamma)m^c}{(4c+2\gamma-6)T + (2c+1-\gamma)m^c X}. \end{aligned}$$

Therefore, the number of solutions of (139) does not exceed the number of roots of non-zero polynomial of degree at most 3.

On the other hand, from (116), (120), (124) and (131) – (133) we have

$$Y''(l) \ll vqmN^{-\gamma} \quad \text{and} \quad Y'''(l) \ll vqm^2N^{-2\gamma}. \tag{140}$$

Hence, we conclude that the interval $(L_1, L_2]$ can be divided into at most 13 intervals such that if J is one of them, then at least one of the following assertions holds:

$$|Y''(l)| \asymp vqmN^{-\gamma} \quad \text{uniformly for} \quad l \in J, \tag{141}$$

$$|Y'''(l)| \asymp vqm^2N^{-2\gamma} \quad \text{uniformly for} \quad l \in J. \tag{142}$$

If (141) holds, then we use (4), (105) and Van der Corput's theorem (see [8, Chapter 1, Theorem 5]) for the second derivative to get

$$\begin{aligned} \sum_{l \in J} e(Y(l)) &\ll L(qvmN^{-\gamma})^{\frac{1}{2}} + (qvmN^{-\gamma})^{-\frac{1}{2}} \\ &\ll q^{\frac{1}{2}}v^{\frac{1}{2}}LM^{\frac{1}{2}}N^{-\frac{\gamma}{2}} + q^{-\frac{1}{2}}v^{-\frac{1}{2}}M^{-\frac{1}{2}}N^{\frac{\gamma}{2}}. \end{aligned} \tag{143}$$

In the case when (142) is satisfied we apply (4), (105) and Van der Corput's theorem for the third derivative to get

$$\begin{aligned} \sum_{l \in J} e(Y(l)) &\ll L(qvm^2 N^{-2\gamma})^{\frac{1}{6}} + L^{\frac{1}{2}}(qvm^2 N^{-2\gamma})^{-\frac{1}{6}} \\ &\ll q^{\frac{1}{6}} v^{\frac{1}{6}} L M^{\frac{1}{3}} N^{-\frac{\gamma}{3}} + q^{-\frac{1}{6}} v^{-\frac{1}{6}} L^{\frac{1}{2}} M^{-\frac{1}{3}} N^{\frac{\gamma}{3}}. \end{aligned} \quad (144)$$

Hence, in each case, $\sum_{l \in J} e(Y(l))$ can be estimated by the sum of the expressions on the right sides of the inequalities (143) and (144). Therefore, we obtain

$$\sum_{L_1 < l \leq L_2} e(Y(l)) \ll q^{\frac{1}{2}} v^{\frac{1}{2}} L M^{\frac{1}{2}} N^{-\frac{\gamma}{2}} + q^{-\frac{1}{2}} v^{-\frac{1}{2}} M^{-\frac{1}{2}} N^{\frac{\gamma}{2}} + q^{\frac{1}{6}} v^{\frac{1}{6}} L M^{\frac{1}{3}} N^{-\frac{\gamma}{3}} + q^{-\frac{1}{6}} v^{-\frac{1}{6}} L^{\frac{1}{2}} M^{-\frac{1}{3}} N^{\frac{\gamma}{3}}.$$

We use (4), (105), (111) and (113) and find that

$$W_{M,L} \ll N^\varepsilon \left(N^\gamma Q^{-\frac{1}{2}} + v^{\frac{1}{4}} Q^{\frac{1}{4}} N^{\frac{7\gamma}{8}} + v^{-\frac{1}{4}} Q^{-\frac{1}{4}} N^{\frac{7\gamma}{8}} + v^{\frac{1}{12}} Q^{\frac{1}{12}} N^{\frac{11\gamma}{12}} + v^{-\frac{1}{12}} Q^{-\frac{1}{12}} N^{\frac{23\gamma}{24}} \right).$$

We apply the above estimate for $W_{M,L}$ in (130). Then, by (34) and (55) we obtain

$$\begin{aligned} \Omega_{4,3} &\ll N^\varepsilon \sum_{d \leq D} \sum_{h \leq H} \frac{1}{h} \sum_{r < A_1 \log^3 N} \left(N^\gamma Q^{-\frac{1}{2}} + \left(\frac{h}{d} \right)^{\frac{1}{4}} Q^{\frac{1}{4}} N^{\frac{7\gamma}{8}} \right. \\ &\quad \left. + \left(\frac{h}{d} \right)^{-\frac{1}{4}} Q^{-\frac{1}{4}} N^{\frac{7\gamma}{8}} + \left(\frac{h}{d} \right)^{\frac{1}{12}} Q^{\frac{1}{12}} N^{\frac{11\gamma}{12}} + \left(\frac{h}{d} \right)^{-\frac{1}{12}} Q^{-\frac{1}{12}} N^{\frac{23\gamma}{24}} \right) \\ &\ll N^\varepsilon \left(D N^\gamma Q^{-\frac{1}{2}} + D Q^{\frac{1}{4}} N^{\frac{1}{4} + \frac{5\gamma}{8}} + D^{\frac{5}{4}} Q^{-\frac{1}{4}} N^{\frac{7\gamma}{8}} + D Q^{\frac{1}{12}} N^{\frac{1}{12} + \frac{5\gamma}{6}} + D^{\frac{13}{12}} Q^{-\frac{1}{12}} N^{\frac{23\gamma}{24}} \right). \end{aligned}$$

With the choice of Q which we made in (128) it is now clear that

$$\Omega_{4,3} \ll \frac{N^{2\gamma-1}}{(\log N)^2}. \quad (145)$$

Now, let us carry on with the study of $\Omega_{4,1}$. We have choosen the constant α_1 in such a way, that from (76) and (122) it follows that $Y''(l) \asymp vqm N^{-\gamma}$ uniformly for $l \in (L_1, L_2]$. Then the sum $\sum_{\frac{P}{m} < l \leq \frac{2P}{m}} e(Y(l))$ can be bounded by the expression on the right side of (143). This observation illustrates that $\Omega_{4,1}$ is bounded by the same quantity as $\Omega_{4,3}$, i.e.

$$\Omega_{4,1} \ll \frac{N^{2\gamma-1}}{(\log N)^2}. \quad (146)$$

In a very similar manner one can show that

$$\Omega_{4,2} \ll \frac{N^{2\gamma-1}}{(\log N)^2}. \quad (147)$$

Then, from (121), (129) and (145) – (147) we get

$$\Omega_4 \ll \frac{N^{2\gamma-1}}{(\log N)^2}. \quad (148)$$

It remains to find a bound for Ω_3 . The same argument as the one for Ω_4 can be applied here once more to show that

$$\Omega_3 \ll \frac{N^{2\gamma-1}}{(\log N)^2}. \quad (149)$$

From (102), (103), (148) and (149) we conclude that (73) is satisfied and the theorem is proved.

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